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RESEARCH ARTICLE

Exponential stability of bioprocess model with mass balance and sequential reactor systems

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Abstract

This study investigates the stability of a bioprocess model that incorporates mass balance principles within a sequential reactor system equipped with a recycling loop. The analysis employs one-dimensional partial differential equations, including integro-differential terms, to evaluate exponential stability via a Port-Hamiltonian control mechanism. The proposed approach demonstrates how a simplified structural framework, used to estimate the carbon dehydration process within a dynamic model, enables seamless integration ofcarbon transfer control strategies across domains. Numerical examples are included to illustrate the model's effectiveness and real-world applicability.

Keywords: boundary port variables; exponential stability; semigroup theory

Introduction

Growing environmental concerns and global sustainability goals have intensified the need for innovative solutions to manage solid waste, particularly food and electronic waste. A port-Hamiltonian system was recently developed using computational system to identify carbon content and its composition and degradation behavior (1, 2). Bioprocess systems have emerged as effective frameworks for processing such waste through energy-efficient, adaptive techniques. The boundary port variable method is one of popular methods in physical network theory, enables a modular representation of complex process systems (3). In this study, we presented a stability analysis of a bioprocess model that integrates mass balance principles with a sequential reactor configuration featuring a recycling loop. This model employs a one-dimensional partial differential equation (PDE) with integro-differential terms to accurately describe the dynamic behavior of the system under varying boundary conditions (4). The integro-differential forms with the controller provide an estimate of carbon dehydration and the tolerable rate of carbon transfer between tankers.

Bioprocess systems have emerged as effective frameworks for processing such waste through energy-efficient, adaptive techniques (5). This approach, known as port-based modeling, provides a unified method for analyzing multiple physical domains, including mechanical, electrical, hydraulic and thermal systems. It offers a systematic framework for modeling complex physical processes. Continuous stirred-tank bioreactors are often used to explore nonlinear adaptive control strategies in anaerobic depollution processes. By using boundary port variables and Dirac structures, the system is equipped with a boundary control mechanism that preserves energy exchange and enhances operational robustness. The design of linear input-output estimators play a crucial role in maintaining exponential stability and ensuring predictable system behavior under fluctuating load conditions. Adaptive control mechanisms, including integrated state observers and parameter estimation, are also introduced using nonlinear process models (6). These mechanisms act as software sensors that allow online monitoring of biological conditions and help parameterize variables in dehydration techniques.

The sequential batch reactor-based wastewater treatment processes using advanced control strategy is currently a major area of practical interest (7). A primary motivation in engineering is to apply control techniques to improve system stability and production efficiency (8). Stability analysis of heat exchangers with delayed boundary feedback have studied in (9). However, few control models have been applied to measurement of liquid-gas oxygen transfer rate tanker to tanker (10) and the decomposition of food or electronic waste materials (11). The stability of Bioprocess model with linear case recycle loop has developed in (12). Bioprocess kinetics presents an advanced control method for biotechnology-based wastewater treatment solutions. This study addresses this gap by proposing a recursively structured, carbon-responsive control model capable of adapting to varying input loads.

Port-Hamiltonian modeling with adaptive control system is particularly effective for processing food or electronic waste materials dehydration, ionic polymer metal composites (13, 14). It allows an adaptive fashion to dehydration and fermentation techniques used in using anoxic carbon- based fluidized bed reactor (14). Prior applications of this modeling approach include adaptive control of stirred-tank bioreactors, where software sensors and parameter estimators have been employed for online monitoring and optimization (15, 16). Recent work has discussed adaptive control has focused on despite progress in process control technologies, limited work has addressed the stabilization of biodiesel process models dealing explicitly with heterogeneous waste streams (16). Food and electronic waste differ significantly in thermal properties, moisture content and carbon composition, demanding a flexible and scalable approach to treatment. This problem is addressed using software sensors (3), we introduced and delineated the carbon measurement process under dynamic model and it readily seamless carbon transfer integration of control strategies across domains, including biochemical reaction to be tolerable the tanker, liquid-gas oxygen transfer rate tanker to tanker.

The exponential stability and feedback control of the linearized basic bioprocess model are discussed using the spectrum -determined growth assumption (17). Similar approaches are found in⊠(18-24), which include analysis of the counter-flow heat exchange equation under zero boundary conditions, as well as border feedback stability. In this paper, we discuss the stability of carbon transfer rate measurement process (transfer rate tanker to tanker) that incorporates mass balance and a sequential reactor-type recycling loop, governed by integro-differential equation with Port-Hamiltonian control mechanism (25). The effectiveness of the proposed system is demonstrated through numerical simulations that validate the theoretical findings and showcase the model's practical applicability (26-28). The approach provides a foundation for future integration with sensor-based monitoring and intelligent control systems for scalable waste treatment operations, including biochemical reaction to be tolerable the tanker, carbon transfer rate tanker to tanker.

The paper is organized as follows: Section 2 is devoted to the modeling of an anaerobic digestion bioprocess with recycle loop of one-dimensional partial differential equations. Some exponential stability and basic lemmas are proposed in section 3 and in section 4, the proposed results are verified both by analytically and numerically. Conclusion will be given in section 5.

Formation of problem and basic notations

Consider bioprocess model of sequential reactor with carbon dehydration measurement between tankers (aerator+settler mass balance equation) is (5, 8, 29, 30).

An anaerobic biological wastewater treatment process is

$$\frac{\partial}{\partial t}x_1(t,z)=-\frac{\partial}{\partial z}x_1(t,z)-\frac{F_{in}+F_r}{V}x_1(t,z)-a_{11}x_1(t,z)-a_{12}x_2(t,z)$$

$$\frac{\partial}{\partial t}x_2(t,z)=-\frac{\partial}{\partial z}x_2(t,z)-\frac{F_{in}+F_r}{V}x_2(t,z)-a_{21}x_1(t,z)-a_{22}x_2(t,z)-a_{23}x_2(t,z)$$

$$\frac{\partial}{\partial t}x_3(t,z)=-\frac{\partial}{\partial z}x_3(t,z)-\frac{F_{in}+F_r}{V}x_3(t,z)+a_{31}x_1(t,z)+a_{32}x_2(t,z)-a_{33}x_3(t,z),$$

$$(t,z)\in(0,\infty)x[0,1],$$

$$x_1(t,0)=u_1(t),\quad x_2(t,0)=u_2(t),\ x_3(t,0)=u_3(t),\ t\in(0,\infty),$$

$$x_1(0,z)=x_{10}(z),\ x_2(0,z)=x_{20}(z),\ x_3(0,z)=x_{30}(z),\ z\in[0,1],$$

$$x_1(t,1)=y_1(t),\quad x_2(t,1)=y_2(t),\ x_3(t,1)=y_3(t),\ t\in(0,\infty).$$

described in (11, 24). This system typically operates alongside a sedimentation tank (settler), which separates solids from the liquid and an aerator, where biological degradation of contaminants occurs. This arrangement allows for the separation of biomass from the treated wastewater. Excess biomass is removed from the system, while part of the settled biomass is recycled back into the bioreactor. The variables $x_1(t, z)$ denote the influent substrate concentration, $x_2(t, z)$ the oxygen feed rate and $x_3(t, z)$the concentration of recycled biomass, respectively. The process can be mathematically represented as a one-dimensional partial differential equation as follows;

$$\frac{\partial}{\partial t}x(t,z)=-G_1\frac{\partial}{\partial z}\left(x(t,z)+\int_a^b F(t,z)x(t,z)dt\right)+G_0x(t,z) \qquad 1$$

u(t) = Bx(t, z), x(0, z) = $x_0(z)$, t × z ϵ (0, ∞) × [0,1],

y(t) = Cx(t, z), t × z ϵ (0, ∞) × [0,1],

where the state $x(t,z)=\begin{pmatrix}x_1(t,z)\\x_2(t,z)\\x_3(t,z)\end{pmatrix}$ is take the values in R^3, $F(t, b) = F_{in}$ and $F(t, a) = F_r$ is influent of recycle of waste flow rates and V is an aerator and settler volume respectively. Let B, C, G_0, G_1, F be an $n \times n$ matrix, $G_1=G_1^T$ satisfies and V = 1,then

$$G_1=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\ \text{and}\ G_0=\begin{pmatrix}-a_{11}&-a_{12}&0\\-a_{21}&-a_{22}&-a_{32}\\a_{31}&a_{32}&-a_{33}\end{pmatrix}$$

The term denotes $\int_a^b\frac{\partial}{\partial z}F(t,z)x(t,z)dt$ carbon-responsive control, which is recycling back into the bioreactor. The input variable $u(t)=\begin{pmatrix}u_1(t)\\u_2(t)\\u_3(t)\end{pmatrix}\in R^3$ represents the amount of carbon, quantified based on equivalent weights of food or electronic waste to be processed. Accordingly, we assume that the output $y(t)=\begin{pmatrix}y_1(t)\\y_2(t)\\y_3(t)\end{pmatrix}\in R^3$, is inked to the system's output feedback $u_1(t) = k_1y_1(t)$, $u_2(t) = k_2y_2(t)$ and $u_3(t) = k_3y_3(t)$. The state variables are equivalent to $x_1(t,0) = k_1x_1(t,1)$, $x_2(t,0) = k_2x_2(t,1)$, $x_3(t,0) = k_3x_3(t,1)$, t ϵ (0,∞) and k_1, k_2, k_3 are some feedback gain constant. However, the process yields a higher output for a given input while maintaining effective biological degradation of carbon contaminants.

Remark 1

Anaerobic digestion is a complex bioprocess that can be modelled as a dynamical system. From an engineering perspective, a suitable reduced-order model can be used to represent this system effectively. The singular perturbation method offers one possible systematic approach for its control. Some of the products generated in this process exhibit extremely low solubility. It is also important to note that the quantified carbon input corresponds to the equivalent weight of food or electronic waste materials processed by the two microorganisms $F(t, b) = F_{in}$ and $F(t, a) = F_r$.

Remark 2

[1] Consider the differential operator J of order N such that

$$J\varepsilon_i=\sum_{i=0}^{N}M_i(i)\frac{d^n}{dz^i}\left(\varepsilon_i(z)+\int_a^b F(t,z)\varepsilon_i(t,z)dt\right), \qquad (2)$$

Where $\varepsilon_i \in C^{\infty}((a, b): R^N)$ and $M_1(i)$, (i=1,2, ….., N) is an $n \times n$ real matrix. If $J\varepsilon_i$ is skew symmetric then can be written as follows

$$<\varepsilon_1, J\varepsilon_2>+<\varepsilon_2, J\varepsilon_1>=0.$$

To verify the validity and stability of recycle loop for Eqn. (1), the following lemmas are necessary.

Materials and Methods

Exponential stability

In this section, we demonstrate how Eqn. (1) can be applied to stability analysis. Before presenting the main result, we introduce several foundational lemmas. These lemmas show that the port-Hamiltonian variables are linear combinations of the boundary values, with a one-to-one correspondence.

Lemma 1

Let $\xi_i \in H^N((a, b): R^n)$, i = 1,2 be an N times differentiable on the Sobolve space at the interval $(a; b)$ then for any J as in remark 2,

$$\int_a^b \left[\xi_1^T(z)(J\xi_2)(z)+\xi_2^T(z)(J\xi_1)(z)\right]dt = \left(\xi_1^T(z)\ldots\frac{d^{N-1}}{dz^{N-1}}\xi_1^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi_1^T(z)\right)A \times \begin{pmatrix}\xi_2(z)\\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}}\xi_2^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi_2^T(z)\end{pmatrix} \quad (3)$$

Furthermore, Λ is a symmetric matrix such that $A_{ij}=A$

$$A_{ij}=\begin{cases}0, & i+j>N,\\ M_1(k)(-1)^{i-1}, & i+j-1=k.\end{cases}$$

Proof: Let us choose J be as in remark 1,

$$\int_a^b \left[e_1^T(z)(J\xi_2)(z)+\xi_2^T(z)(J\xi_1)(z)\right]dt = \sum_{i=1}^{N}\int_a^b\left[\xi_1^T(z)M_1(i)\frac{d^i}{dz^i}\left(\xi_2(z)+\int_a^b F(t,z)\xi_2(z)dt\right) + \xi_2^T(z)M_1(i)\frac{d^i}{dz^i}\left(\xi_1(z)+\int_a^b F(t,z)\xi_1(z)dt\right)\right]ds. \quad (4)$$

We assume that the restriction of our proof for $i \geq 1$,

$$\int_a^b \left[e_1^T(z)(J\xi_2)(z)+\xi_2^T(z)(J\xi_1)(z)\right]dt = \sum_{i=1}^{N}\int_a^b\left[\xi_1^T(z)M_1(i)\frac{d^i}{dz^i}\left(\xi_2(z)+\int_a^b F(t,z)\xi_2(z)dt\right) + \xi_2^T(z)M_1(i)\frac{d^i}{dz^i}\left(\xi_1(z)+\int_a^b F(t,z)\xi_1(z)dt\right)\right]ds. \quad (5)$$

If $M_1(i) = M_1(i)^T$, by using integration by parts on the right-hand side of the above equation,

$$\begin{aligned}\int_a^b \left[\xi_1^T(z)(J\xi_2)(z)+\xi_2^T(z)(J\xi_1)(z)\right]dt &= \sum_{i=1}^{N}\Bigg[(-1)^j\xi_1^T(z)M_1(i)\frac{d^{i-1}}{dz^{i-1}}\left(\xi_2(z)+\int_a^b F(t,z)\xi_2(z)dt\right)\\ &+(-1)^j\xi_2^T(z)M_1(i)\frac{d^i}{dz^i}\left(\xi_1(z)+\int_a^b F(t,z)\xi_1(z)dt\right)\Bigg]_a^b\\ &+(-1)^j\int_a^b\frac{d}{dz}\xi_1(z)\frac{d^{i-1}}{dz^{i-1}}\left(\xi_2(z)+\int_a^b F(t,z)\xi_2(z)dt\right)ds\\ &+(-1)^j\int_a^b\frac{d}{dz}\xi_2(z)\frac{d^{i-1}}{dz^{i-1}}\left(\xi_1(z)+\int_a^b F(t,z)\xi_1(z)dt\right)ds\end{aligned}$$

After some manipulation and put $i - 1 - j = k$,

$$\begin{aligned}&\int_a^b\left(\xi_1^T(z)\frac{d^i}{dz^i}\left(\xi_2(z)+\int_a^b F(t,z)\xi_2(z)dt\right)+\xi_2^T(z)\frac{d^i}{dz^i}\left(\xi_1(z)+\int_a^b F(t,z)\xi_1(z)dt\right)\right)ds\\ &\leq\Bigg[\sum(-1)^j\frac{d^j}{dz^j}\xi_1^T(z)M_1(i)\times\Bigg[\frac{d^{i-1}}{dz^{i-1}}(\xi_2(z))\\ &+\frac{d^{i-2-j}}{dz^{i-2-j}}((F(t,b)-F(t,a)))(\xi_2(z)dz)\Bigg]\Bigg]_a^b.\end{aligned}$$

The above inequality shows that, for any skew symmetric differential operator J gives rise to a symmetric bilinear product on the space of boundary conditions

$$\Lambda\left[\xi(a)\cdots\frac{d^{N-1}}{dz^{N-1}}\xi(a)+\frac{d^{N-2}}{dz^{N-2}}(F(t,b)-F(t,a))\xi(a),\ \xi(b)\cdots\frac{d^{N-1}}{dz^{N-1}}\xi(b)+\frac{d^{N-2}}{dz^{N-2}}(F(t,b)-F(t,a))\xi(b)\right]$$

The coefficient of this symmetric product, where captured in the matrix Λ are uniquely defined by the coefficients of the skew-symmetric differential operator J.

Therefore

$$\Lambda=\begin{pmatrix}M_1(1) & M_1(2) & M_1(3) & \cdots & M_1(N-1) & M_1(N)\\ -M_1(2) & -M_1(3) & -M_1(4) & \cdots & -M_1(N) & 0\\ M_1(3) & M_1(4) & \ddots & \ddots & 0 & 0\\ -M_1(4) & \ddots & \ddots & \ddots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{N-1}-M_1(N) & 0 & \cdots & \cdots & \cdots & 0\end{pmatrix}.$$

Lemma 2

Let $H^N(a, b)$: R^n denote the sobolev space of N times differentiable functions on the interval (a, b): The subspace D_j of A defined as

$$D_j(A)=\left\{\begin{pmatrix}f_\partial\\ \xi_\partial\end{pmatrix}\middle|\ \xi\in H^N((a,b):R^n), J_\xi=G_1\left(\frac{\partial}{\partial z}\xi(z)+\int_a^b\frac{\partial}{\partial z}F(t,z)\xi(z)dt\right)=f,\ \begin{pmatrix}f_\partial\\ \xi_\partial\end{pmatrix}=R_{ext}\begin{pmatrix}\xi(b)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}\xi(b)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi(b)\\ \xi(a)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}\xi(a)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi(a)\end{pmatrix}\right\}$$

is a Dirac structure.

Proof: Let power of every element in $D_j(A)$ is zero,

$$\begin{aligned}&<(f,f_\partial,\xi,\xi_\partial),(f,f_\partial,\xi,\xi_\partial)>=<\xi,f>L^2+<f,\xi>L^2-<\xi_\partial,f_\partial>L^2-<f_\partial,\xi_\partial>L^2\\ &=<\xi,f>L^2+<f,\xi>L^2-2\xi_\partial^T f_\partial\\ &=\int_a^b\Bigg[\xi_1^T(z)G_1\left(\frac{\partial}{\partial z}\xi_2(z)+\int_a^b\frac{\partial}{\partial z}F(t,z)\xi_2(z)dt\right)\\ &\quad+\xi_2^T(z)G_1\left(\frac{\partial}{\partial z}\xi_1(z)+\int_a^b\frac{\partial}{\partial z}F(t,z)\xi_1(z)dt\right)\Bigg]ds-2\xi_\partial^T f_\partial\\ &=\left[\xi^T(z),\cdots,\frac{d^{N-1}}{dz^{N-1}}\xi^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi^T(z)\right)\Lambda\\ &\quad\times\left[\begin{pmatrix}\xi^T(z)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}\xi^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi^T(z)\end{pmatrix}\right]_a^b-2\xi_\partial^T f_\partial\\ &=\left(f_\partial^T,\xi_\partial^T\right)\sum\begin{pmatrix}f_\partial\\ \xi_\partial\end{pmatrix}-2\xi_\partial^T f_\partial\\ &=\left(\xi^T(z),\cdots,\frac{d^{N-1}}{dz^{N-1}}\xi^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi^T(z)\right)\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}\\ &\quad\times\left[\begin{pmatrix}\xi^T(z)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}\xi^T(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}\xi^T(z)\end{pmatrix}\right]_a^b-2\xi_\partial^T f_\partial\\ &=\left(f_\partial^T,\xi_\partial^T\right)\sum\begin{pmatrix}f_\partial\\ \xi_\partial\end{pmatrix}-2\xi_\partial^T f_\partial=0,\end{aligned}$$

using Lemma 1 we can concluded that D_J is a Dirac structure.

Remark 3: The matrix Λ_{ext} is in $R^{2nN\times 2nN}$ and associated with differential operator J; then

$$\Lambda_{ext}=\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix},\quad R_{ext}=\frac{1}{\sqrt{2}}\begin{pmatrix}\Lambda & -\Lambda\\ I & I\end{pmatrix}\ and\ \begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}=R_{ext}^T\sum R_{ext}$$ hold,

where $\sum=\begin{pmatrix}\Lambda & 0\\ 0 & -\Lambda\end{pmatrix}$.

Lemma 3

Let U be a full rank matrix of size $2nN \times k$. If there exists, the operator A and $D(A)$;

$$A\xi = J\xi \tag{6}$$

where

$D(A)$ ={$\xi \in L^2((a,b): R^n|$ The port variables are associate to $\xi, \begin{pmatrix} f_\partial \\ \xi_\partial \end{pmatrix}$, $f \in L^2((a,b): R^n)$ and $(f, f_\partial, \xi, \xi_\partial) \in D_J$}

If adjoint A is equal to - f, then

$D(A^*)$ ={$\xi \in L^2((a,b): R^n|$ The port variables are associate to $\xi, \begin{pmatrix} f_\partial \\ \xi_\partial \end{pmatrix}$, is in ker $U^T\Sigma \in L^2((a, b): R^n)$}

Proof: Let ξ_1 be an element of L^2 ((a, b): R^n) and its domain is $D(A^*)$ if and only if $\overline{\xi_1}$ and $\xi_2 \in D(A)$,

$$<\xi_1, A\xi_2> = <\xi_1, \xi_2> \tag{7}$$

Where

$$A^*\xi_2 = \overline{\xi_1}. \tag{8}$$

Let $f = G_1 \frac{\partial}{\partial z}\left(x(t,z) + \int_a^b F(t,z)x(t,z)dt\right) \in H^1((a,b): R^n)\ and\ \xi_1 \in D(A)$, we compute

$$< \xi_1, A\xi_2 > = \int_a^b \xi_1^T(z)(J\xi_2)(z)dt. \tag{9}$$

It is well known that, for every function in $H^N((a, b): R^n)$ to be zero at the boundaries is in the domain of A,

$$\begin{aligned} < \xi_1, A\xi_2 > &= \int_a^b \xi_1^T(z)(J\xi_2)(z)dt + \int_a^b \xi_2^T(z)(J\xi_1)(z)dt - \int_a^b \xi_2^T(z)(J\xi_1)(z)dt \\ &= \int_a^b \xi_1^T(z)\left[G_1 \frac{\partial}{\partial z}\left(\xi_2(t,z) + \int_a^b F(z)\xi_2(t,z)dt\right)\right]ds \\ &+ \int_a^b \xi_2^T(z)\left[G_1 \frac{\partial}{\partial z}\left(\xi_1(t,z) + \int_a^b F(z)\xi_1(t,z)dt\right)\right]ds \\ &- \int_a^b \xi_2(z)^T(I\xi_1)(z)dt. \end{aligned} \tag{10}$$

By using Lemma 2,

$$< \xi_1, A\xi_2 > = (\xi_{\partial,1}^T, \xi_{\partial,1}^T)\Sigma\begin{pmatrix} f_{\partial,2} \\ \xi_{\partial,2} \end{pmatrix} - \int_a^b \xi_2(z)^T(J\xi_1)(z)dt. \tag{11}$$

Here $\begin{pmatrix} f_{\partial,1} \\ \xi_{\partial,1} \end{pmatrix}$ and $\begin{pmatrix} f_{\partial,2} \\ \xi_{\partial,2} \end{pmatrix}$ are port boundary variables associat- with ξ_1 and ξ_2, then $\begin{pmatrix} f_{\partial,2} \\ \xi_{\partial,2} \end{pmatrix} = U_r$ for some $r \in R^k$ and apply this equation in (9),

$$< \xi_1, A\xi_2 > = (\xi_{\partial,1}^T, \xi_{\partial,1}^T)\Sigma U_r + \int_a^b \xi_2(z)^T(-J\xi_1)(z)dt. \tag{12}$$

Using condition (11) in (12), we can conclude that $\begin{pmatrix} f_{\partial,1} \\ \xi_{\partial,1} \end{pmatrix} \in \ker(U^T\Sigma)$ and $A^*\xi_1 = -J\xi_1$.

Lemma 4

Let J_P is an infinitesimal generator of contraction semigroup and its domain $D(J_P)$. If there exists, the operator P has a full rank matrix and satisfies $P\Sigma P^T \geq 0$ such that

$$J_P\xi = J\xi$$

where

$D(J_P)$ = {$\xi \in L^2((a, b): R^n|$ The port variable associated to $\xi, \begin{pmatrix} f_\partial \\ \xi_\partial \end{pmatrix}$ is in ker P and $f \in L^2((a, b): R^n), (f, f_\partial, \xi, \xi_\partial) \in D_J$

Proof: The operator J_P is an infinitesimal generator of a contraction semi group if and only if P has a rank nN and satisfies $P\Sigma P^T \geq 0$. The proof is smaller to Lemma 3. From that Lemma 3, $\begin{pmatrix} f_\partial \\ \xi_\partial \end{pmatrix}$ lies within the kernel of $U^T\Sigma$ and we have that

$(\ker U^T\Sigma)^T\Sigma(\ker U^T\Sigma) \geq 0$, since Σ has nN positive eigen values (see (1)).

Therefore, we conclude that A is an infinitesimal generator of a contraction semigroup and combining the matrix P and U, since the kernel of P is equals to the image of U, so we can define that $U_2^TU_1 + U_1^TU_2 \leq 0\left(\text{"where" } U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}\right)$ Using Lemma 3, one proves easily

$$D(A) = \left\{x \in H^1((a,b): R^n) \middle| \begin{pmatrix} f_{\partial,x(t)} \\ \xi_{\partial,x(t)} \end{pmatrix} \in \ker P\right\}. \tag{13}$$

Next, we examine feedback stabilization for a scattering energy-preserving system. In this context, the closed-loop system is modelled as a boundary control system, where the operator A generates a contraction semigroup. Our goal is to determine whether the closed-loop system exhibits exponential stability.

Definition 3.1

Assume that $x \in H^1((a, b): R^N)$. If there exists a boundary port variable are associated with the system (1) and the vector $(f_{\partial,x}, \xi_{\partial,x}) \in R^n$,

$$\begin{pmatrix} f_\partial \\ \xi_\partial \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \Xi & \Xi \\ I & I \end{pmatrix}\begin{pmatrix} x(b) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}}x(b) + \frac{d^{N-2}}{dz^{N-2}}(F(t,b) - F(t,a))x(b) \\ x(a) \\ \vdots \\ \frac{d^{N-1}}{dz^{N-1}}x(a) + \frac{d^{N-2}}{dz^{N-2}}(F(t,b) - F(t,a))x(a) \end{pmatrix}, \tag{14}$$

where Ξ is an $n \times n$ matrix (which is defined below). Furthermore, let P_1 and P_2 be an $n \times 2n$ matrices and associated with Σ

("where" $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ in R^n and $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ such that

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}\Sigma\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}^T = \begin{pmatrix} P_1\Sigma P_1^T & P_1\Sigma P_2^T \\ P_2\Sigma P_1^T & P_2\Sigma P_2^T \end{pmatrix} \tag{15}$$

is invertible and if and only if $\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ is also invertible,

Where

$$\Xi = \begin{pmatrix} G_1(1) & G_1(2) & G_1(3) & \cdots & G_1(N-1) & G_1(N) \\ -G_1(2) & -G_1(3) & -G_1(4) & \cdots & -G_1(N) & 0 \\ G_1(3) & G_1(4) & \ddots & \ddots & 0 & 0 \\ -G_1(4) & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{N-1} - G_1(N) & 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$$

Theorem 3.1

Assume that P_1 be an $n \times 2n$ matrix. If there exists a matrix P has full rank and satisfies $P\Sigma P^T \geq 0$ (" where" $\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ in R^n), such that the system (1) is associated with input $u(t) = P\begin{pmatrix} f_{\partial,x(t)} \\ \xi_{\partial,x(t)} \end{pmatrix}$.

Furthermore, if the infinitesimal generator A exists

$$Ax=\left\{G_1\frac{\partial}{\partial z}\left(x(t,z)+\int_a^b\frac{\partial}{\partial z}F(t,z)x(t,z)dt\right)+G_0x(t,z)\right\},$$

with domain

$$D(A)=\left\{x\in H^1((a,b):R^n)\middle|\begin{pmatrix}f_{\partial,x(t)}\\ \xi_{\partial,x(t)}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\Xi & \Xi\\ I & I\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}x(z)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}x(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}x(z)\end{pmatrix} is\ in\ rank\ P\right\},$$

and the adjoint of A is equal to $-G_1\frac{\partial}{\partial z}\left(x(t,z)+\int_a^b\frac{\partial}{\partial z}F(t,z)x(t,z)dt\right)$ and

$$D(A^*)=\left\{x\in H^1((a,b):R^n)\middle|\begin{pmatrix}f_{\partial,x(t)}\\ \xi_{\partial,x(t)}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\Xi & \Xi\\ I & I\end{pmatrix}\right.$$
$$\left.\times\begin{pmatrix}x(z)\\ \vdots\\ \frac{d^{N-1}}{dz^{N-1}}x(z)+(F(t,b)-F(t,a))\frac{d^{N-2}}{dz^{N-2}}x(z)\end{pmatrix} is\ in\ rank\ P^T\Sigma\right\},$$

then D(A) and D(A^*) generates a contraction semigroup on X,

Proof: If P be a full rank of n × 2n matrix and $\begin{pmatrix}P_1\\ P_2\end{pmatrix}$ is invertible, then $\Xi_{P_1P_2}$ can be written as follows

$$\Xi_{P_1P_2}=\begin{pmatrix}P_1\Sigma P_1^T & P_1\Sigma P_2^T\\ P_2\Sigma P_1^T & P_2\Sigma P_2^T\end{pmatrix}^{-1}. \qquad (16)$$

Let C be an operator associated with the system (1), then C is mapping from $C: L^{-1}(H^1(a,b): R^n)\to R^n$,

$$Cx(t)=P\begin{pmatrix}f_{\partial,x(t)}\\ \xi_{\partial,x(t)}\end{pmatrix}$$

and the output

$$y(t)=Cx(t)$$

If $u\in C^2(0,\infty: R^k)$ and as per Lemma 2,

$$\frac{1}{2}\frac{d}{dt}\|x(t)\|_L^2=\frac{1}{2}\begin{pmatrix}u^T & y^T\end{pmatrix}\Xi_{P_1P_2}\begin{pmatrix}u\\ y\end{pmatrix}-<\int_a^b\frac{\partial}{\partial z}F(t,z)x(t,z)dt>-G_0x(t),\ x(t)>.$$

Moreover, port-controlled Hamiltonian systems are described in terms of their energy balance rate, which is determined by specific system dynamics. This energy balance plays a crucial role in establishing exponential stability results for boundary control systems. In recent years, several researchers have shown interest in the study of boundary control systems (12, 18).

Theorem 3.2

Assume that Lemma 1-4 and Theorem 3.1 hold. If there exist u(t) = 0, for all 0 ≤ τ ≤ t, the energy system $S(t)=\frac{1}{2}\|x(t)\|_L^2$ satisfies large enough on such $\Xi_{P_1P_2}$ that

$S(\tau)\le C(\tau)\int_0^T\|x(t,a)\|_R^2\,dt$ and $S(\tau)\le C(\tau)\int_0^T\|x(t,b)\|_R^2\,dt$,

where C is a positive constant that is depending on τ and the system (1) is exponential stable when

$\frac{d}{dt}S(\tau)\le -k_1\|x(t,a)\|_R^2$ and $\frac{d}{dt}S(\tau)\le -k_2\|x(t,b)\|_R^2$

where k_1, k_2 are positive constants.

Proof: The energy system is

$$S(t)=\frac{1}{2}\int_a^b x^T(t,z)\Gamma(z)x(t,z)dt, \qquad (17)$$

where Γ(z) is energy balancing constant, it could be less than one. To prove the inequality, we need to employ the result (21-23),

$$K(z)=\int_\theta^{\tau-\theta}x^T(t,z)\Gamma(z)x(t,z)dt, \qquad (18)$$

Where $\beta>0$, $\tau>\beta=2(b-a)$ and for all $z\in[a,b]$.

Therefore

$$K'(z)=\int_\theta^{\tau-\theta}x^T(t,z)\Gamma(z)\frac{\partial}{\partial z}x(t,z)dt+\int_\theta^{\tau-\theta}(\frac{\partial}{\partial z}x(t,z)^T\Gamma(z)x(t,z))dt \qquad (19)$$
$$+\beta x^T(\theta,z)\Gamma(z)x(\theta,z)+\beta x^T(\tau-\theta,z)\Gamma(z)x(\tau-\theta,z).$$

Since G_1, G_0 are non-singular n × n matrices, then

$$\frac{\partial}{\partial t}x(t,z)=-G_1\frac{\partial}{\partial z}\left(x(t,z)+\int_a^b F(t,z)x(t,z)dt\right)+G_0x(t,z) \qquad (20)$$

Substituting equation (20) into (19)

$$K'(z)=\int_\theta^{\tau-\theta}x^T(t,z)\Gamma(z)(\frac{\partial}{\partial t}x(t,z)dt+G_1\int_\theta^{\tau-\theta}\frac{\partial}{\partial z}F(t,z)^Tx(t,z)dt+G_0\mathrm{x(t,z))ds}$$
$$+\int_\theta^{\tau-\theta}(\frac{\partial}{\partial t}x(t,z)+G_1\int_a^b\frac{\partial}{\partial z}(F(t,z)x(t,z)dt+G_0x^T(t,z)\Gamma(z)x(t,z)ds+\beta x^T(t,z)\Gamma(z)x(t,z)$$
$$+\beta x^T(\tau-\theta,z)\Gamma(Z)x(\tau-\theta,z)=-x^T(t,z)\Gamma(z)G_1x(t,z)\Big|_{z=\theta}^{z=\tau-\theta}+\int_\theta^{\tau-\theta}x^T\Gamma(z)[F(t,b)x(t,b)$$
$$-F(t,a)x(t,a)+(F(t,b)x(t,b)-F(t,a)x(t,a))^T+G_0x(t,z)]dt+\beta x^T(\theta,z)\Gamma(z)x(\theta,z)$$
$$+\beta x^T(\tau-\theta,z)\Gamma(z)x(\tau-\theta,z)$$

Since

$$M^*=Max\left\{G_0,(F(t,b)(x(t,b))-F(t,a)x(t,a)+(F(t,b)(x(t,b))-F(t,a)x(t,a))^T\right\},$$
$$K'(z)=-x^T(t,z)G_1^{-1}\Gamma(z)x(t,z)\Big|_{z=0}^{z=\tau-\theta}+\int_\theta^{\tau-\theta}x^T(t,z)M^*\Gamma(z)x(t,z)dt$$
$$+\beta x^T(\theta,z)\Gamma(z)x(\theta,z)+\beta x^T(\tau-\theta,z)\Gamma(z)\times x(\tau-\theta,z).$$

By choosing β is large enough and that Γ(z)<1,

$$K'(z)\ge -x^T(t,z)G_1^{-1}\,x(t,z)\Big|_{z=0}^{z=\tau-\theta}-\int_\theta^{\tau-\theta}x^T(\theta,z)M^*x(\theta,z)dt.$$

Using Grownall's inequality and that τ > 2(b - a),

$$K(z)\le e^{-M^*(b-a)}\|x(t,b)\|^2.$$

If τ - β > 2(b - a), without loss of generality we may assume that above inequality holds and

$$(\tau-2\beta)S(\tau)\le(\tau-2\beta)S(\tau-2\beta)$$
$$\le\frac{1}{2}\int_a^b\int_\theta^{\tau-\theta}x^T(t,z)\Gamma(z)x(t,z)dt\,ds$$
$$\le\int_a^b e^{-M^*(b-a)}\|x(s,b)\|^2dt$$
$$S(\tau)\le\frac{(b-a)}{2(\tau-2\beta)}e^{-M^*(b-a)}\int_0^\tau\|x(t,b)\|^2\,dt.$$

Since we observe that it is sufficient to prove the existence of some time τ > 0 and some constant $C_0=\frac{(b-a)}{2(\tau-2\beta)}e^{-M^*(b-a)}>0$ then

$$S(\tau)\le C_0\int_0^\tau\|x(s,b)\|^2\,dt \qquad (21)$$

is solution of system (1). Indeed, assume that $S(0)=\frac{C_0}{1+C_0},\ \forall C_0\le 1,$

$$S(\tau)-S(0)+S(0)\le C_0\int_0^{\tau}\|x(s,b)\|_R^2\,dt. \quad (22)$$

From equation (22) if $C_1 = Max\{S(0), C_0\}$ we can deduce that

$$\frac{dS(\tau)}{dt}\le -C_1\|x(t,b)\|_R^2 \quad (23)$$

and system (1) is exponentially stable with respect of semigroup operator J_P, Next, we will find a relation between $\|x(t, b)\|$ and y(t), Using Theorem 3.1,

$$\begin{pmatrix}0\\ y\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}P_1\\ P_2\end{pmatrix}\begin{pmatrix}\Xi & \Xi\\ I & I\end{pmatrix}\begin{pmatrix}x(t,b)\\ x(t,a)\end{pmatrix} \quad (24)$$

Since Ξ and $\begin{pmatrix}P_1\\ P_2\end{pmatrix}$ are non-singular matrix and also N^* is invertible such that $N^*\|x\|^2 \ge \epsilon\|x\|^2$ for all $\epsilon>0$. Therefore taking norm on both sides for the above equation

$$\|y\|^2 = N^*\|x(t, b)\|^2 \ge \epsilon\|x(t, b)\|^2,$$

Where $N^*=\frac{1}{\sqrt{2}}\begin{pmatrix}P_1\\ P_2\end{pmatrix}\begin{pmatrix}\Xi & \Xi\\ I & I\end{pmatrix}$ This implies $\|x(t, b)\|^2 \le \epsilon^{-1}\|y\|^2$ and hence proved theorem

Results and Discussion

Examples

In this section, we demonstrate how to apply the results from the previous section. By appropriately selecting the input (i.e., boundary conditions) and the output, we show that a simple matrix condition can be used to establish exponential stability.

Example 4.1

(Bioprocess model): Consider the following bioprocess control model

$$\frac{\partial}{\partial t}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\left[\frac{\partial}{\partial z}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\int_a^b\frac{\partial}{\partial z}\begin{pmatrix}F_1 & F_2 & F_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}dt\right]+\begin{pmatrix}-a_{11}&-a_{12}&0\\-a_{21}&-a_{22}&-a_{23}\\-a_{31}&-a_{32}&-a_{33}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \quad (25)$$

Based on Theorem 3.1 and 3.2 our approach can be applied for linear transformation, we select the state variable as

$$\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}=\psi^{-1}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\ with\ \psi=\begin{pmatrix}\sqrt{a_{11}}&0&0\\0&\sqrt{a_{22}}&0\\0&0&\sqrt{a_{33}}\end{pmatrix}. \quad (26)$$

From Eqn. (26) we can find matrix $\psi^{-1}P_\psi$.

Therefore

$$\psi^{-1}P\psi=\begin{pmatrix}-a_{11}&-a_{12}\frac{\sqrt{a_{12}}}{\sqrt{a_{11}}}&0\\-a_{21}\frac{\sqrt{a_{11}}}{\sqrt{a_{22}}}&-a_{22}&-a_{32}\frac{\sqrt{a_{33}}}{\sqrt{a_{22}}}\\-a_{31}\frac{\sqrt{a_{11}}}{\sqrt{a_{33}}}&-a_{32}\frac{\sqrt{a_{22}}}{\sqrt{a_{33}}}&-a_{33}\end{pmatrix} \quad (27)$$

From (25), the integral function can be derived as

$$\int_a^b\frac{\partial}{\partial z}\begin{pmatrix}F_1 & F_2 & F_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}dt=\big((F_1(b)-F_1(a))\ \ (F_2(b)-F_2(a))\ \ (F_3(b)-F_3(a))\big)\times\begin{pmatrix}\sqrt{a_{11}}&0&0\\0&\sqrt{a_{22}}&0\\0&0&\sqrt{a_{33}}\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}. \quad (28)$$

Substituting Eqn. (26-28) in (25),

$$\frac{\partial}{\partial t}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}=-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\left[\frac{\partial}{\partial z}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}+\big((F_1(b)-F_1(a))\ \ (F_2(b)-F_2(a))\ \ (F_3(b)-F_3(a))\big)\times\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix}\right]+\begin{pmatrix}a_{11}&a_{12}\frac{\sqrt{a_{12}}}{\sqrt{a_{11}}}&0\\a_{21}\frac{\sqrt{a_{11}}}{\sqrt{a_{22}}}&a_{22}&a_{32}\frac{\sqrt{a_{33}}}{\sqrt{a_{22}}}\\-a_{31}\frac{\sqrt{a_{11}}}{\sqrt{a_{33}}}&-a_{32}\frac{\sqrt{a_{22}}}{\sqrt{a_{33}}}&-a_{33}\end{pmatrix}\begin{pmatrix}\theta_1\\\theta_2\\\theta_3\end{pmatrix} \quad (29)$$

and the boundary condition is

$$\begin{aligned}k_1\sqrt{a_{11}}x_1(b)+\sqrt{a_{11}}x_1(a)&=0\\k_2\sqrt{a_{22}}x_2(b)+\sqrt{a_{22}}x_2(a)&=0\\k_3\sqrt{a_{33}}x_3(b)+\sqrt{a_{33}}x_3(a)&=0\end{aligned} \quad (30)$$

Therefore, the boundary conditions are associated with boundary port-variables, which can write as

$$\begin{pmatrix}f_{\partial,x}\\\xi_{\partial,x}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}&\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\\\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}&\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\end{pmatrix}\times\begin{pmatrix}x_1(b)&x_1(a)\\x_2(b)&x_2(a)\\x_3(b)&x_3(a)\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}(x_1(b)-x_1(a))\\(x_2(b)-x_2(a))\\(x_3(b)-x_3(a))\\(x_1(b)+x_1(a))\\(x_2(b)+x_2(a))\\(x_3(b)+x_3(a))\end{pmatrix}. \quad (31)$$

To obtain exponential stability with boundary port-variables, we can select the matrix Q as follows

$$Q=\frac{1}{\sqrt{2}}\begin{pmatrix}a_{11}&-k_1\sqrt{a_{12}}&0&\sqrt{a_{11}}&k_1\sqrt{a_{12}}&0\\-k_2\sqrt{a_{21}}&a_{22}&\sqrt{a_{23}}&k_2\sqrt{a_{21}}&\sqrt{a_{22}}&\sqrt{a_{23}}\\\sqrt{a_{31}}&-k_3\sqrt{a_{32}}&-a_{33}&\sqrt{a_{31}}&k_3\sqrt{a_{32}}&\sqrt{a_{33}}\end{pmatrix}. \quad (32)$$

Let Q be an operator, which relates to the equation (1) and generates a contraction semigroup satisfies the condition (by using Theorem 3.1)

$$\int_a^b\frac{\partial F(z)}{\partial z}dz+Q\Sigma Q^T=\begin{pmatrix}(F_1(b)-F_1(a))&0&0\\0&(F_2(b)-F_2(a))&0\\0&0&(F_3(b)-F_3(a))\end{pmatrix}+\begin{pmatrix}a_{11}^{3/2}-k_1^2a_{12}&0&0\\0&-k_2^2a_{21}+a_{22}^{3/2}+a_{23}&0\\0&0&-k_3^2a_{32}-a_{33}^{3/2}+a_{31}\end{pmatrix}\ge 0 \quad (33)$$

and

$$k_1^2 \le \frac{(F_1(b)-F_1(a))+a_{11}^{3/2}}{a_{12}}$$
$$k_2^2 \le \frac{(F_2(b)-F_2(a))+a_{22}^{3/2}+a_{23}}{a_{21}} \quad (34)$$
$$k_3^2 \le \frac{(F_3(b)-F_3(a))+a_{33}^{3/2}+a_{31}}{a_{32}}$$

By using Eqn. (31-34) in (29),

$$A_T\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = -\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\left[\frac{\partial}{\partial z}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\int_a^b\frac{\partial}{\partial z}\begin{pmatrix}F_1 & F_2 & F_3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}dt\right]$$
$$+\begin{pmatrix}-a_{11} & -a_{12}\frac{\sqrt{a_{12}}}{\sqrt{a_{11}}} & 0\\ -a_{21}\frac{\sqrt{a_{11}}}{\sqrt{a_{22}}} & -a_{22} & -a_{32}\frac{\sqrt{a_{33}}}{\sqrt{a_{22}}}\\ -a_{31}\frac{\sqrt{a_{11}}}{\sqrt{a_{33}}} & -a_{32}\frac{\sqrt{a_{22}}}{\sqrt{a_{33}}} & a_{33}\end{pmatrix}\times\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

with domain

$$D(A_T)=\left\{x\in H^1(a,b)\left|\begin{pmatrix}f_{\partial,x}\\ \xi_{\partial,x}\end{pmatrix}\in\ker Q\right.\right\}$$

generates a contraction semigroup. This is turn implies that the corresponding system (1), which generates uniformly bounded and the infinitesimal generator A_T is satisfied C_0 semigroup, therefore the system (25) is exponentially stable.

Example 4.2

To verify the validity of recycle loop, we executed the numerical values. In this example, it shows that how much inputs can be added to the bioprocess model during the time interval [0,1] and it could be stabilized. We set as

$a_{11} = 0.1, a_{12} = 0.9, a_{13} = 0, a_{21} = 0.7, a_{22} = 0.2$

$a_{23} = 0.1, a_{31} = 0.1, a_{32} = 0.8, a_{33} = -0.3$

and $k_1 = k_2 = k_3 = 1$ are tolerable carbon transfer rate between tankers, then existing Q matrix is given in last page. Therefore $Q\Sigma Q^T \ge 0$, however, Theorem 3.2, the stability will depend on the choice of input and the system (29) with boundary of the value (30) is corresponding to the system (25). The following table shows that how much of input values can be tolerated the output values and shows how much error occurs (if $k_1 = k_2 = k_3 = 1$). It shows Table 1, Table 2 and Table 3.

The boundary condition is exists in [0, 1] and the inputs are $(F_3(b) - F_3(a)) = u_3(t)$, it shows that input values from $(F_3(b) - F_3(a)) = 0.5$ to $(F_3(b) - F_3(a)) = 1.0$ is stable and the remaining values are unstable, if and only if less than 0.5 does not satisfies $k_3^2 \le \frac{(F_3(b)-F_3(a))+a_{31}-a_{33}^{3/2}}{a_{32}}$ and also input values greater than 1 is invalid, because boundary values in [0 1]. If we consider $(F_2(b) - F_2(a)) = u_2(t)$, the input values from $(F_2(b) - F_2(a)) = 0.5$ to $(F_2(b) - F_2(a)) = 1.0$ is stable, then $(F_1(b) - F_1(a)) = u_1(t)$ and stability is exist at $(F_1(b) - F_1(a)) = 0.5$ to $(F_1(b) - F_1(a)) = 1.0$. In particular, if we choose the boundary conditions are exist in [-10,10] and $(F_3(b) - F_3(a)) = u_3(t)$, $(F_2(b) - F_2(a)) = u_2(t)$, $(F_1(b) - F_1(a)) = u_1(t)$, simulating k_1, k_2, k_3 in the ration of 0.1, the response of $u_i(t), i = 1, 2, 3$ is showed in Fig 1. The response of error $e_i(t) = u_i(t) - y_i(t)$ (where $y_i(t) = CX_i(t, 1)$ for i=1,2,3 and C is $n \times n$ are showed in Fig 2. In [14] has taken into an account of $k_1 = k_2 = k_3 = 1$, which is indicated as recycle rates are maintained as a constant value. But in our methodology, we have considered different values for $(F_i(b) - F_i(a))$, i = 1,2,3 while the recycle rates are associated with carbon transfer between tankers.

Table 1. The error value of $k_3^2 \le \frac{(F_3(b)-F_3(a))+a_{31}-a_{33}^{3/2}}{a_{32}}$

Input (F3(b) - F_3 (a))	Output $k_3^2 \le \frac{(F_3(b)-F_3(a))+a_{31}-a_{33}^{3/2}}{a_{32}}$	error
0.5000	1.06891	0.06891
0.6000	1.08039	0.08039
0.7000	1.02053	0.02053
0.8000	1.03303	0.03307
0.9000	1.04553	0.04553
1.0000	1.04803	0.04801

Table 2. The error value of $k_2^2 \le \frac{(F_2(b)-F_2(a))+a_{23}+a_{22}^{3/2}}{a_{21}}$

Input (F2(b) - F_2(a))	Output $k_2^2 \le \frac{(F_2(b)-F_2(a))+a_{23}+a_{22}^{3/2}}{a_{21}}$	error
0.5000	1.05634	0.05634
0.6000	1.03277	0.03277
0.7000	1.02706	0.02706
0.8000	1.04934	0.04934
0.9000	1.05563	0.05563
1.0000	1.05589	0.0558

Table 3. The error value of $k_1^2 \le \frac{(F_1(b)-F_1(a))+a_{11}^{3/2}}{a_{12}}$

Input (F1(b) - F_1(a))	Output $k_1^2 \le \frac{(F_1(b)-F_1(a))+a_{11}^{3/2}}{a_{12}}$	error
0.5000	1.07634	0.07634
0.6000	1.05262	0.05262
0.7000	1.03706	0.03706
0.800	1.02803	0.02803
0.9000	1.05513	0.05513
1.0000	1.05462	0.05462

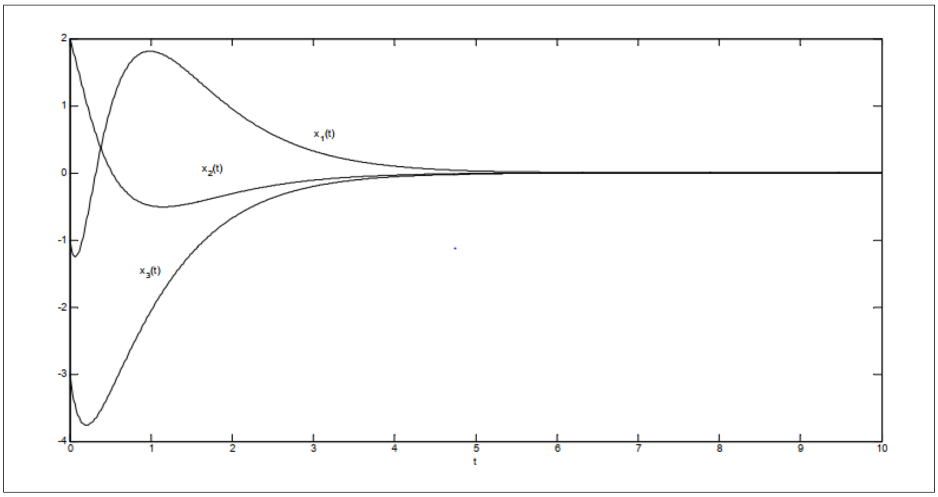

Fig. 1. The response of x_1, x_2, x_3.

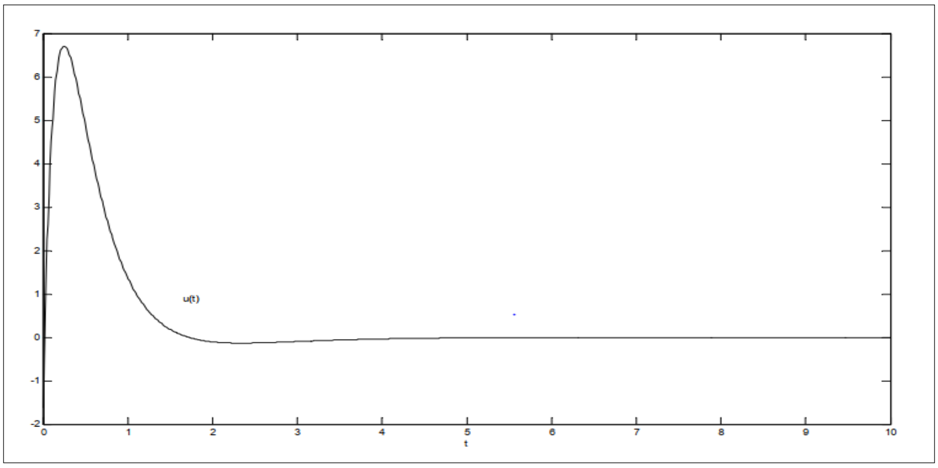

Fig. 2. The response of u(t).

$$Q = \frac{1}{\sqrt{2}}\begin{bmatrix} 0.316227 & -0.948683 & 0 & 0.316227 & 0.948683 & 0 \\ -0.836660 & 0.447213 & 0.316227 & 0.836660 & 0.447213 & 0.316227 \\ 0.316227 & -0.447213 & 0.836660 & -0.316227 & -0.447213 & -0.836660 \end{bmatrix}$$

Conclusion

This study analyzed the stability of a bioprocess model that integrates mass balance concepts with a sequential reactor system and recycling loop, using one-dimensional partial differential equations with integro-differential terms. Exponential stability was examined through a Port-Hamiltonian control framework, in which boundary damping conditions applied to input and output variables facilitated stabilization of the system's energy storage function. The structural framework supports carbon dehydration measurement within a dynamic model, while the controller maintains a tolerable carbon transfer rate between tankers.

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Authors' contributions

RRK carried out the anaerobic depollution processes theoretical studies, participated in the sequence alignment and drafted the manuscript. TA and VG are carried out wastewater treatment employs biotechnology solutions using bioprocess kinetics. SS carried out modeling of an anaerobic digestion bioprocess. SM carried out sequential reactor type with boundary port control mechanism. KS developed MATLAB code for bioprocess model examples, sequence alignment and drafted. All authors read and approved the final manuscript.

Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest

Ethical issues: None

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